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COMPOSITES WITH SYMMETRY AND THEIR EXTREMAL PROPERTIES

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Abstract—We identify microstructures that extremize sums of strain energy densities. For one energy we exhibit optimal laminar composites that are orthotropic with axes given by the eigenbasis of the homogeneous strain. Some partial results on the uniqueness of the optimal microstructures are obtained.

1. INTRODUCTION

The identification of extremal elastic composites has direct application to problems in material science and structural optimization. There has been much work addressing the optimal design of load bearing structures made from two or more elastic materials [see Allaire and Kohn (1993); Jog *et al.* (1993); Bendsøe and Kikuchi (1988); Cherkaev and Gibianskii (1984)]. The problem considered in this paper arises from optimality conditions inherent in the problem of the optimal layout of two elastic materials in a prescribed domain. Here one of the materials is an expensive material with desired stiffness properties, while the other is a less expensive and more compliant material. For a given set of applied loads, the object is to determine the optimal arrangement of the two elastic materials necessary to maximize the overall stiffness. This optimization is carried out subject to a resource constraint on the stiff material. It is well known from theory and numerical experiments [see Murat and Tartar (1985); Lurie *et al.* (1982); Cheng and Olhoff (1981)] that arbitrarily fine mixtures of the two constituents may appear in the optimal design. The problem is made well-posed by extending the design space to include effective elastic tensors corresponding to mixtures of the constituent materials [see Murat and Tartar (1985); Lurie *et al.* (1982)]. Within this extended set of designs the controls are the local volume fraction of the stiff material and an associated effective elastic tensor consistent with the local anisotropy of the mixture. When the local volume fraction is either one or zero, the associated region is occupied by the stiff or compliant material, respectively. Intermediate values of the volume fraction correspond to regions occupied by mixtures of the component materials with elastic properties described by effective elastic tensors. When mixtures occur in the optimal design, the local anisotropy of the mixture is selected to maximize the local strain energy density [see Lurie *et al.* (1982); Kohn (1991)]. For fixed local volume fraction this optimality condition can be stated as follows: for a given strain $\varepsilon(x)$ the effective elastic tensor $C^e(x)$ in the optimal layout maximizes the strain energy density

$$C^e(x)\varepsilon(x)\cdot\varepsilon(x). \tag{1}$$

For problems with multiple independent load cases. One has that the effective elastic tensor $C^e(x)$ maximizes the sum of strain energy densities

$$\sum_{i=1}^{N} C^e(x)\varepsilon^i(x)\cdot\varepsilon^i(x), \tag{2}$$

where ε^i, $i = 1 \ldots, N$ are the local strains corresponding to each load case. In the event that the load cases are specified statistically we may replace the sum with an ensemble average [see Lipton (1994)]:

$$\langle C^e(x)\varepsilon\cdot\varepsilon\rangle. \tag{3}$$

Since $C^e(x)$ is deterministic we may take it outside the average and write

$$\langle C^e\varepsilon\cdot\varepsilon\rangle = \sum_{ijkl} C^e_{ijkl}\langle\varepsilon_{ij}\varepsilon_{kl}\rangle. \tag{4}$$

In this paper we investigate the extremal effective tensors maximizing local strain energy densities of the type (1)–(4). We consider two and three dimensional anisotropic mixtures made from two well-ordered isotropic elastic materials in specified volume fractions. We establish that the extremal effective tensor will inherit all rotational symmetry present in the sum of strain energy densities Motivated by eqns (3) and (4) we rewrite eqn (2) as:

$$tr(C^eM) \equiv \sum_{ijkl} C^e_{ijkl}M_{ijkl}, \tag{5}$$

where

$$M_{ijkl} = \sum_{s=1}^{N} \varepsilon^s_{ij}\varepsilon^s_{kl}. \tag{6}$$

Here the strains have been written out in terms of their components, i.e. $\varepsilon^s = \varepsilon^s_{ij}$. It is easily seen that M_{ijkl} is positive definite and $M_{ijkl} = M_{jikl} = M_{klij}$. For the random case the right hand side of eqn (6) is replaced with the ensemble average $\langle\varepsilon_{ij}\varepsilon_{kl}\rangle$. In what follows we show that the extremal effective tensor inherits whatever rotational symmetry is present in the tensor M formed from the ensemble of local strains.

Mathematically our problem becomes one of characterizing composites with effective elastic tensors C^e that extremize the form $tr\ (C^eM)$ when the tensor M is invariant under a prescribed group of rotations. We show that this form is extremized by finite rank laminar composites with effective tensors invariant under the prescribed rotation group, see Theorem 3.1. In other words there exist extremal composites with crystallographic symmetry consistent with the rotation group.

The scope of Theorem 3.1 includes the case when M is a projection onto a subspace of strains invariant under a prescribed group of rotations, see Corollary 3.1. As a first example, we choose M to be the projection onto constant shears denoted by P_s. Since this subspace is invariant under the group of proper rotations O^3_+, it follows from Corollary 3.1 that there exists isotropic finite rank laminates that extremize $tr\ (C^eM)$, see Section 3. In Section 4 we show that among laminates only the isotropic ones are extremal for $tr\ (C^eP_s)$.

A second case concerns the problem of extremizing the effective strain energy density associated with a prescribed homogeneous strain. We apply Theorem 3.1 to find optimal laminar composites that are orthotropic with axes given by the eigenbasis of the homogeneous strain, see Section 3. Based on this observation we show that the stress tensor is simultaneously diagonal with the homogeneous strain in the optimal composite. We remark that the orthotropy of optimal laminar composites was illucidated earlier by Jog *et al.* (1993) for the two dimensional case. For the cases of two dimensional elasticity and three dimensional incompressible elasticity, the simultaneous diagonalization of stress and strain may be established using optimality conditions on the polarization field in the extremal composite [see Allaire and Kohn (1993); Kohn and Lipton (1988)].

Last, we note that closed form descriptions for the effective elastic tensors of finite rank laminates have been found for the isotropic, cubic, transversely isotropic and orthotropic cases, [see Avellaneda (1987a); Francfort and Murat (1986); James *et al.* (1990); Lipton (1991); Lipton (1993a)]. These descriptions can be used to compute the extremal values and microstructures associated with the form $tr(C^e M)$ when M possesses the requisite symmetry.

2. FINITE RANK LAMINATES

In this section we review the extremality and convexity properties of effective elastic tensors associated with finite rank laminar microstructures.

We consider laminates made from two well-ordered isotropic elastic components in specified volume fraction, the component elasticities are specified by C_i, $i = 1,2$, given by

$$C_i = 2\mu_i \mathscr{I} + (\kappa_i - \tfrac{2}{3}\mu_i) I \otimes I \tag{7}$$

with $\mathscr{I}$ being the identity on 3×3 symmetric matrices and I the 3×3 identity matrix we adopt the convention $\mu_1 \leqslant \mu_2$, $\kappa_1 \leqslant \kappa_2$. The volume fraction of each material is specified by θ_1 for material 1 and θ_2 for material 2, such that $\theta_1 + \theta_2 = 1$.

A finite rank laminate is defined iteratively. To illustrate, we show how to construct a rank 2 laminate. One starts with a core of material 2 and layers it with a coating of material 1 in layers of thickness ε^2 perpendicular to a specified direction n_1. One then takes this finely layered material and again layers it with a coating of material 1 in layers of thickness ε perpendicular to a second direction n_2. The $\varepsilon \to 0$ limit of this microgeometry is called a rank 2 laminate. Conversely one could start with a core of material 1 and layer it with a coating of material 2 and so on. Laminates of higher rank are constructed in the same way. Explicit formulas have been developed for tensors describing the effective properties of finite rank laminates [see Francfort and Murat (1986); Lurie and Cherkaev (1984); Tartar (1985)]. For fixed volume fractions θ_1 and θ_2 of materials 1 and 2 the effective elasticity tensors of a rank j stiff laminate $\overline{C}$ with material 1 as core and material 2 as layers with layer direction given by the unit vectors $n^1, n^2, \ldots, n^j$ is given by

$$\overline{C}(\theta_2; \mathscr{T}_2) \equiv C_2 - (1-\theta_2)[(C_2 - C_1)^{-1} - \theta_2 \mathscr{T}_2]^{-1}. \tag{8}$$

The effective elasticity of a rank j compliant laminate $\underline{C}$ with material 2 as core and material 1 as layers with layering directions $n^1, n^2, \ldots, n^j$ is given by

$$\underline{C}(\theta_2; \mathscr{T}_1) = C_1 + \theta_2[(C_2 - C_1)^{-1} + (1-\theta_2)\mathscr{T}_1]^{-1}. \tag{9}$$

Here

$$\mathscr{T}_r = \sum_{i=1}^{j} \rho_i \hat{\Gamma}^r(n^i), \quad r = 1,2, \tag{10}$$

$$0 \leqslant \rho_i \leqslant 1, \quad \sum_{i=1}^{j} \rho_i = 1, \tag{11}$$

and the tensor $\hat{\Gamma}^r(v)$ is given by

$$\hat{\Gamma}^r_{mnop}(v) = \frac{1}{4\mu_i}(v_m v_o \delta_{np} + v_m v_p \delta_{no} + v_n v_o \delta_{mp} + v_n v_p \delta_{mo}) + \left(\frac{3}{3\kappa_i + 2\mu_i} - \frac{1}{\mu_i}\right) v_m v_n v_o v_p, \tag{12}$$

for all symmetric 3×3 matrices M and $r = 1, 2$. The quantities $(1-\theta_2)\rho_i$ and $\theta_2 \rho_i$ appearing in eqns (8)–(10) are the relative proportions of layer materials introduced in the ith lamination. The geometry of the laminate is encoded in the geometric tensors $\mathscr{T}_r$. To emphasize the dependence of the effective properties on volume fraction and geometric tensor we have written $\overline{C} = \overline{C}(\sigma\theta_2;\ \mathscr{T}_2)$, $\underline{C} = \underline{C}(\theta_2;\ \mathscr{T}_1)$. Formulas (8) and (9) were developed by Francfort and Murat (1986).

We introduce the convex sets of tensors Δ_1, Δ_2 formed by all convex combinations of the type $\mathscr{T}_1$, $\mathscr{T}_2$ delivered by formula (10). To understand the geometry of these sets we regard $\hat{\Gamma}^r(v)$, $r = 1, 2$ given by eqn (12) as tensor valued maps transforming the surface of the unit sphere into surfaces in the space of fourth order totally symmetric tensors. It is evident from eqn (10) that Δ_1 and Δ_2 correspond to the closed convex hulls of these surfaces.

From the convexity of Δ_r, $r = 1, 2$ it follows that any convex combination of geometric tensors $\mathscr{T}_r$ must lie in Δ_r and correspond to the geometric tensor for some finite rank laminar microgeometry.

Finite rank laminates are known to possess extremal elastic properties [see Cherkaev and Gibianskii (1984); Lurie *et al.* (1982); Francfort and Murat (1986); Avellaneda (1987a, b); Kohn and Lipton (1988); Milton and Kohn (1988)]. We let G_{θ_2} be the set of all effective elastic tensors C^e associated with composites made from well-ordered isotropic components C_1 and C_2 in the volume fractions θ_1 and θ_2, respectively. It was demonstrated by Avellaneda (1987b), that for any positive definite fourth order tensor M one has

$$\max_{C^e \varepsilon G_{\theta_2}} tr(C^e M) = tr(\overline{C}M), \tag{13}$$

$$\min_{C^e \varepsilon G_{\theta_2}} tr(C^e M) = tr(\underline{C}M), \tag{14}$$

where the extremal effective tensor $\overline{C}$ appearing in eqn (13) is associated with a stiff laminar composite and the extremal effective tensor $\underline{C}$ appearing in eqn (14) is associated with a compliant laminar composite.

One also has a convexity property for finite rank laminates in terms of the geometric tensor [see Lipton (1994b)]. Indeed, given two geometric tensors $\mathscr{T}_1$ and $\mathscr{T}_1'$ in Δ_1, then for any positive definite symmetric fourth order tensor M and $0 \leqslant w \leqslant 1$ one has:

$$w tr(\underline{C}(\theta_2;\mathscr{T}_1)M) + (1-w) tr(\underline{C}(\theta_2;\mathscr{T}_1')M) \geqslant tr(\underline{C}(\theta_2; w\mathscr{T}_1 + (1-w)\mathscr{T}_1')M). \tag{15}$$

Similarly for $\mathscr{T}_2$ and $\mathscr{T}_2'$ in Δ_2 and for any positive definite symmetric fourth order tensor M and $0 \leqslant w \leqslant 1$ one has:

$$w tr(\overline{C}(\theta_2;\mathscr{T}_2)M) + (1-w) tr(\overline{C}(\theta_2;\mathscr{T}_2')M) \leqslant tr(\overline{C}(\theta_2; w\mathscr{T}_2 + (1-w)\mathscr{T}_2')M). \tag{16}$$

We illustrate how rotation matrices act on the effective elasticity tensors of finite rank laminar microstructures. Denoting a rotation matrix by Q, the conjugation of a fourth order elasticity tensor is denoted by

$$Q \otimes QC^e Q \otimes Q \equiv Q_{im} Q_{jn} Q_{ko} Q_{lp} C^e_{mnop}. \tag{17}$$

Noting that the component elasticities are isotropic it follows immediately from formulas (8) and (9) that

$$Q \otimes Q\overline{C}(\theta_2;\mathscr{T}_2) Q \otimes Q = \overline{C}(\theta_2; Q \otimes Q\mathscr{T}_2 Q \otimes Q) \tag{18}$$

and

$$Q \otimes Q\underline{C}(\theta_2;\mathscr{T}_1) Q \otimes Q = \underline{C}(\theta_2; Q \otimes Q\mathscr{T}_1 Q \otimes Q). \tag{19}$$

For future reference we observe that for $\mathcal{T}_r$ given by eqn (10), we have:

$$Q \otimes Q\mathcal{T}_s Q \otimes Q = \sum_{i=1}^{j} \rho_i \hat{\Gamma}^r(Qn^i). \tag{20}$$

In this way we see that the convex sets Δ_r, r = 1, 2 are invariant under the action of O^3_+. Finally, we note that all elements $\mathcal{T}_r$ of Δ_r lie on the hyperplanes given by

$$tr(\mathcal{T}_r P_s) = \frac{2}{3\kappa_r + 4\mu_r} + \frac{1}{\mu_r}, \tag{21}$$

and

$$tr(\mathcal{T}_r P_h) = \frac{1}{3\kappa_r + 4\mu_r}, \tag{22}$$

where P_s and P_h are the projections onto shears and hydrostatic strains, respectively.

3. EXTREMAL PROPERTIES OF SYMMETRIC COMPOSITES

We denote by $\mathcal{G}$ the matrix representation of a rotation group and state the main result of this exposition.

Theorem 3.1. Let M be a positive definite fourth order tensor invariant under the rotation group $\mathcal{G}$, then there exists extremal effective properties $\overline{C}$ and $\underline{C}$ invariant under rotations in $\mathcal{G}$, associated with finite rank stiff and compliant laminates, respectively such that:

$$\max_{C^e \varepsilon G_{\theta_2}} tr(C^e M) = tr(\overline{C}M) \tag{23}$$

$$\min_{C^e \varepsilon G_{\theta_2}} tr(C^e M) = tr(\underline{C}M). \tag{24}$$

The proof of Theorem 3.1 follows from the extremal, convexity, and rotation properties of finite rank laminates as presented in Section 2. In what follows we establish eqn (23) of theorem 3.1 noting that eqn (24) is proved similarly.

We define the average of a fourth order tensor M over the group $\mathcal{G}$ by

$$\langle M \rangle \equiv \int_{\mathcal{G}} Q \otimes QMQ \otimes Q \mathrm{d}H \tag{25}$$

where H is the Haar measure for the group $\mathcal{G}$. If $\mathcal{G}$ is finite we denote the order of the group by v and write

$$\langle M \rangle = \frac{1}{v} \sum_{\gamma=1}^{v} Q^\gamma \otimes Q^\gamma M Q^\gamma \otimes Q^\gamma \tag{26}$$

for elements Q^γ, $\gamma = 1, \ldots, v$ in $\mathcal{G}$.

Given a fourth order tensor M satisfying the hypothesis of Theorem 3.1, then as in eqn (13) there exists an effective tensor $\overline{C}(\theta_2; \mathcal{T}_2)$ associated with a finite rank stiff laminar composite such that:

$$\max_{C^e \epsilon G_{\theta_2}} tr(C^e M) = tr(\bar{C}(\theta_2 ; \mathcal{T}_2)M). \quad (27)$$

Since M is invariant under $\mathcal{G}$ one has $M = \langle M \rangle$ and we obtain

$$tr(\bar{C}(\theta ; \mathcal{T}_2)M) = tr(\bar{C}(\theta_2 ; \mathcal{T}_2)\langle M \rangle) = tr(\langle \bar{C}(\theta_2 ; \mathcal{T}_2) \rangle M). \quad (28)$$

From the rotation property of laminates it follows that

$$\langle \bar{C}(\theta_2 ; \mathcal{T}_2) \rangle = \int_{\mathcal{G}} \bar{C}(\theta_2 ; Q \otimes Q \mathcal{T}_2 Q \otimes Q)\, \mathrm{d}H. \quad (29)$$

Therefore we may apply the concavity property (16) to the right hand side of eqn (28) to obtain the inequality

$$tr(\langle \bar{C}(\theta_2 ; \mathcal{T}) \rangle M) \leqslant tr(\bar{C}(\theta_2 ; \langle \mathcal{T}_2 \rangle)M). \quad (30)$$

Finally applying eqn (20) and noting that Δ_2 is a convex 14 dimensional set we see from Carathéodory's theorem that

$$\langle \mathcal{T}_2 \rangle = \overset{*}{\mathcal{T}}_2, \quad (31)$$

where $\overset{*}{\mathcal{T}}_2$ is the geometric tensor for a finite rank stiff laminate of the form given by eqn (10). In this way we arrive at the inequality

$$tr(\bar{C}(\theta_2 ; \mathcal{T}_2)M) \leqslant tr(\bar{C}(\theta_2 ; \langle \mathcal{T}_2 \rangle)M) = tr(\bar{C}(\theta_2 ; \overset{*}{\mathcal{T}}_2)M), \quad (32)$$

thus the effective tensor $\bar{C}(\theta_2, \overset{*}{\mathcal{T}}_2)$ is extremal, i.e.

$$\max_{C^e \epsilon G_{\theta_2}} tr(C^e M) = tr(\bar{C}(\theta_2 ; \overset{*}{\mathcal{T}}_2)M). \quad (33)$$

To see that $\bar{C}(\theta_2, \overset{*}{\mathcal{T}}_2)$ is invariant under $\mathcal{G}$ we note that for any rotation Q in $\mathcal{G}$ one has

$$Q \otimes Q \bar{C}(\theta_2, \overset{*}{\mathcal{T}}_2) Q \otimes Q = \bar{C}(\theta_2, Q \otimes Q \overset{*}{\mathcal{T}}_2 Q \otimes Q) = \bar{C}(\theta_2, \overset{*}{\mathcal{T}}_2). \quad (34)$$

The first equality follows from eqn (18) and the last equality follows from the fact that $\overset{*}{\mathcal{T}}_2$ is the group average of the tensor $\mathcal{T}_2$ and therefore invariant under $\mathcal{G}$.

We consider the action of a rotation group $\mathcal{G}$ on the space of 3×3 strain matrices $S^{3 \times 3}$. The invariant subspaces are denoted by V_1, V_2, . . . , V_3 and their associated projections are given by $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_j$.

An immediate consequence of Theorem 3.1 is the following corollary.

Corollary 3.1. Given a projection $\mathcal{P}_i$ onto an invariant subspace of a rotation group $\mathcal{G}$ then there exists extremal effective properties $\bar{C}$ and $\underline{C}$ invariant under rotations in $\mathcal{G}$, associated with finite rank stiff and compliant laminates, respectively, such that:

$$\max_{C^e \epsilon G_{\theta_2}} tr(C^e \mathcal{P}_i) = tr(\bar{C} \mathcal{P}_i), \quad (35)$$

$$\min_{C^e \epsilon G_{\theta_2}} tr(C^e \mathcal{P}_i) = tr(\underline{C} \mathcal{P}_i). \quad (36)$$

As a first application we consider the projection $\mathscr{P}_s$ onto the space of shear strains, and find optimal microstructures extremizing $tr(C^e\mathscr{P}_s)$. Since this space is invariant under the group O^3_+ of proper rotations we see from Corollary 3.1 that there exists extremal effective properties $\bar{C}$ and $\underline{C}$ which associated with isotropic finite rank stiff and compliant laminates, respectively. It was shown by Francfort and Murat (1986) that the set of isotropic effective properties delivered by laminar microgeometries consists of just two tensors; one corresponding to compliant laminates and the other to stiff laminates. The layer microgeometries and associated effective properties of isotropic laminates can be found in Francfort and Murat (1986). In Section 4, we strengthen this result and show that among laminates only the isotropic ones extremize $tr(C^eP_s)$.

Next we consider optimal microgeometries with effective tensors extremizing one energy. Given a uniform strain ε, the effective strain energy in a composite is given by $C^e\varepsilon\cdot\varepsilon$. Algebraic manipulation gives

$$C^e\varepsilon\cdot\varepsilon = tr(C^eM), \tag{37}$$

where $M = \varepsilon\otimes\varepsilon$. We expand the strain ε in its spectral resolution

$$\varepsilon = \sum_{i=1}^{3} s_i\delta^i\otimes\delta^i, \tag{38}$$

and consider the "orthotropic" $\mathscr{H}$ of 180 degree rotations around each eigenaxes δ^i, $i = 1$, 2, 3. For the choice $M = \varepsilon\otimes\varepsilon$ one has

$$Q\otimes QMQ\otimes Q = M, \tag{39}$$

for every rotation Q in $\mathscr{H}$.

Therefore from Theorem 3.1 we have that

$$\max_{C^e\varepsilon G_{\theta_2}} tr(C^eM) = tr(\bar{C}M), \tag{40}$$

$$\min_{C^e\varepsilon G_{\theta_2}} tr(C^eM) = tr(\underline{C}M), \tag{41}$$

where $\bar{C}$ and $\underline{C}$ are invariant under rotations in $\mathscr{H}$ and are associated with finite rank stiff and compliant laminates, respectively.

In this way we see that the optimal laminar microgeometry is orthotropic with respect to the eigenaxes of the strain. This observation was made by Jog *et al.* (1993) for the two dimensional case using the results of Pederson (1989) on optimal alignment of orthotropic materials in two dimensional elasticity.

We consider the strain in the optimal composite $\bar{C}$ appearing in eqn (40). The strain σ is given by

$$\sigma = \bar{C}\varepsilon. \tag{42}$$

For any Q in $\mathscr{H}$ we observe that

$$\begin{aligned} Q\sigma Q^T &= Q(\bar{C}\varepsilon)Q^T && (43)\\ &= Q\bar{C}(Q^T\varepsilon Q)Q^T && (44)\\ &= Q\otimes Q\bar{C}Q\otimes Q\varepsilon && (45)\\ &= \bar{C}\varepsilon = \sigma. && (46)\end{aligned}$$

Here the second equality follows from the identity $\varepsilon = Q^{T}\varepsilon Q$ and the second to last equality follows from the invariance of $\bar{C}$ under $\mathscr{H}$. Thus as $Q\sigma Q^{T} = \sigma$ for all Q in $\mathscr{H}$ we see that the stress σ is simultaneously diagonal with the strain ε in the optimal laminar composite.

Identical arguments show that the associated stress in the compliant laminar composite $\underline{C}$ minimizing the elastic energy is simultaneously diagonal with the strain. We note that these results are naturally consistent with the conditions for optimal orientation of orthotropic materials given in Seregin and Troitskii (1982).

4. UNIQUENESS OF EXTREMAL PROPERTIES

It is of interest to examine the uniqueness of effective elastic properties that extremize various measures of structural performance. In this section we consider the uniqueness of effective elastic properties extremizing the functional $tr(C^{e}P_{s})$ introduced in Section 3. There it was shown that extremal effective properties could be found in the class of isotropic effective tensors associated with finite rank laminates. Here we strengthen this result and show that among laminates only the ones with isotropic effective tensors are optimal.

We consider the problem of maximizing $tr(C^{e}P_{s})$ and show only a stiff laminate with an isotropic effective tensor is optimal among laminates. To do this we introduce the function $g(t)$ defined for $0 \leqslant t \leqslant 1$ given by

$$g(t) \equiv tr(\bar{C}(\theta_2\,;t\mathscr{T}_2+(1-t)\overset{*}{\mathscr{T}}_2)P_s), \tag{47}$$

where $\overset{*}{\mathscr{T}}_2$ is the geometric tensor of an isotropic finite rank stiff laminate, and $\mathscr{T}_2$ is a fixed but arbitrary geometric tensor. The set of geometric tensors $\overset{*}{\mathscr{T}}_2$ for stiff isotropic laminates consists of just one tensor [see Francfort and Murat (1986)] and is given by

$$\overset{*}{\mathscr{T}}_2 = \frac{1}{5}\left(\frac{2}{3\kappa_2+4\mu_2}+\frac{1}{\mu_2}\right)P_s+\frac{1}{3\kappa_2+4\mu_2}P_h. \tag{48}$$

We observe that for $t = 0$, one has

$$g(0) = tr(\bar{C}(\theta_2\,;\overset{*}{\mathscr{T}}_s)P_s), \tag{49}$$

where $\bar{C}(\theta_2\,;\overset{*}{\mathscr{T}}_2)$ is the effective tensor of an isotropic stiff laminate shown to be maximal in Section 3. For $t = 1$

$$g(1) = tr(\bar{C}(\theta_2\,;\mathscr{T}_2)P_5), \tag{50}$$

where $\bar{C}(\theta_2\,;\mathscr{T}_2)$ is the effective tensor of a laminate with geometric tensor $\mathscr{T}_2$.

Our claim follows from the strict concavity of the function $g(t)$. Indeed, if $g(t)$ is strictly concave and there exists a stiff laminar composite $\bar{C}(\theta_2\,;\mathscr{T}_2)$ maximizing $tr(C^{e}P_s)$ then

$$\begin{aligned} tr(\bar{C}(\theta_2\,;\mathscr{T}_2)P_s) &= tr(\bar{C}(\theta_2\,;\overset{*}{\mathscr{T}}_2)P_s) \\ &= t\,tr(\bar{C}(\theta_2\,;\mathscr{T}_2)P_s)+(1-t)tr(\bar{C}(\theta_2\,;\overset{*}{\mathscr{T}}_2)P_s) \\ &= t g(1)+(1-t)g(0) < g(t). \end{aligned} \tag{51}$$

Noting that the tensor $\bar{C}(\theta_2\,;t\mathscr{T}_2+(1-t)\overset{*}{\mathscr{T}}_2)$ appearing in the definition of $g(t)$ corresponds to a stiff laminate with geometric tensor

$$\hat{\mathscr{T}}_2 = t\mathscr{T}_2+(1-t)\overset{*}{\mathscr{T}}_2, \tag{52}$$

we arrive at the contradiction

$$tr(\bar{C}(\theta_2\,;\overset{*}{\mathscr{T}}_2)P_s) < tr(\bar{C}(\theta_2\,;\hat{\mathscr{T}}_2)P_s), \tag{53}$$

and our claim follows.

We now establish strict concavity for $g(t)$. For geometric tensors $\mathscr{T}_2$ and $\overset{*}{\mathscr{T}}_2$ we denote their restrictions to the subspace of shears by $\mathscr{T}_{2,s}$ and $\overset{*}{\mathscr{T}}_{2,s}$, respectively. We observe that if

$$\mathscr{T}_{2,s} = \overset{*}{\mathscr{T}}_{2,s}, \tag{54}$$

then from eqn (49) we have

$$\mathscr{T}_{2,s} = \frac{1}{5}\left(\frac{2}{3k_2+4\mu_2}+\frac{1}{\mu_2}\right)P_s, \tag{55}$$

and from eqn (22) it follows that

$$\mathscr{T}_2 = \overset{*}{\mathscr{T}}_2. \tag{56}$$

Motivated by these remarks we need only show the strict concavity of the function $g(t)$ for any choice of geometric tensor $\mathscr{T}_2$ such that $\mathscr{T}_{2,s} \neq \overset{*}{\mathscr{T}}_{2,s}$. Strict concavity of $g(t)$ follows immediately from the observation:

Lemma 4.1. For any geometric tensor $\mathscr{T}_2$ such that $\mathscr{T}_{2,s} \neq \overset{*}{\mathscr{T}}_{2,s}$ one has $\partial_t^2\, g(t) < 0$ for $0 < t < 1$.

Before establishing the lemma, we compute $\partial_t^2\, g(t)$ to obtain:

$$\partial_t^2\, g(t) = -2(1-\theta_2)tr(\{(B-tD)^{-1}D(B-tD)^{-1}D(B-tD)^{-1}\}P_s), \tag{57}$$

where

$$B = (C_2-C_1)^{-1}-\theta_2\overset{*}{\mathscr{T}}_2, \quad \text{and} \quad D = \mathscr{T}_2-\overset{*}{\mathscr{T}}_2. \tag{58}$$

Introducing an orthonormal basis on the space of shears given by ξ^i, $i = 1, \ldots, 5$ and expanding P_s in its spectral representation we find that:

$$\partial_t^2\, g(t) = -2(1-\theta_2)\sum_{i=1}^{5}|(B-tD)^{-1/2}D(B-tD)^{-1}\xi^i|^2 \leqslant 0. \tag{59}$$

It is evident that to establish the lemma it suffices to show: if $\mathscr{T}_{2,s} \neq \overset{*}{\mathscr{T}}_{2,s}$ then $\partial_t^2\, g(t) \neq 0$ for $0 < t < 1$. We establish this by a contrapositive argument; we show that if $\partial_t^2\, g(t) = 0$ for some t in $(0, 1)$ then $\mathscr{T}_{2,s} = \overset{*}{\mathscr{T}}_{2,s}$. Indeed, suppose $\partial_t^2\, g(t) = 0$ for some t in $(0, 1)$, then from eqn (59) it follows that

$$(B-tD)^{-1/2}D(B-tD)^{-1}\xi^i = 0, \quad i = 1,2,\ldots,5. \tag{60}$$

Since $(B-tD)^{-1/2}$ is non-singular it is evident that the vectors $(B-tD)^{-1}\xi^i$ lie in the kernel of D and one has the system of equations:

$$y^i = (B-tD)^{-1}\xi^i, \tag{61}$$

$$Dy^i = 0, \tag{62}$$

for $i = 1, 2, \ldots, 5$.

Multiplying both sides of eqn (61) by $(B-tD)$ and application of eqn (62) gives

$$y^i = B^{-1}\xi^i, \quad i = 1,\ldots,5. \tag{63}$$

Since $\overset{*}{\mathscr{T}}_2$ is isotropic, it follows that the tensor B^{-1} is isotropic and

$$y^i = 2b\xi^i, \tag{64}$$

where $b > 0$ is the shear modulus of the tensor B^{-1}. Multiplying eqn (64) by D and applying eqn (62) gives

$$\mathscr{T}_2\xi^i = \overset{*}{\mathscr{T}}_2\xi^i, \tag{65}$$

for the basis of shears ξ^i, $i = 1, 2, \ldots, 5$, and we conclude $\mathscr{T}_{2,s} = \overset{*}{\mathscr{T}}_{2,s}$.

Last, we remark that similar arguments show that $tr(C^eP_s)$ is minimized only by an isotropic finite rank compliant laminate, i.e. all anisotropic laminates give larger values of $tr(C^eP_s)$. We note that the exclusive minimization of $tr(C^eP_s)$ by finite rank isotropic laminates was first shown using necessary conditions of optimality for the Voigt bounds on polycrystalline elastic composites as done by Milton (1993).

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